88127201

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Candidate session number

0	0							

Tuesday 6 November 2012 (afternoon)

2 hours

Examination code

8	8	1	2	–	7	2	0	1

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

0120

20 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 4]*

Given that $\frac{\pi}{2} < \alpha < \pi$ and $\cos\alpha = -\frac{3}{4}$, find the value of $\sin 2\alpha$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

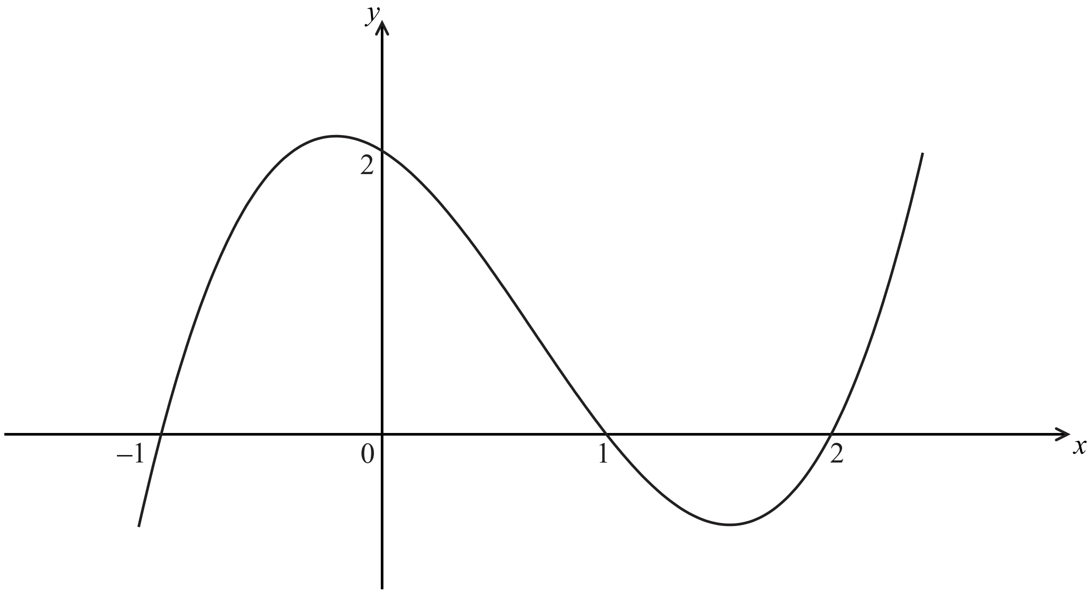
. . . .

2. *[Maximum mark: 4]*

Expand and simplify $\left(\frac{x}{y}-\frac{y}{x}\right)^4$.

3. *[Maximum mark: 7]*

Let $f(x) = x^3 + ax^2 + bx + c$, where a, b, $c \in \mathbb{Z}$. The diagram shows the graph of $y = f(x)$.



(a) Using the information shown in the diagram, find the values of a, b and c. *[4 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

(This question continues on the following page)

(Question 3 continued)

(b) If $g(x) = 3f(x-2)$,

(i) state the coordinates of the points where the graph of g intercepts the x-axis.

. . . .

. . . .

. . . .

. . . .

(ii) Find the y-intercept of the graph of g. *[3 marks]*

. . . .

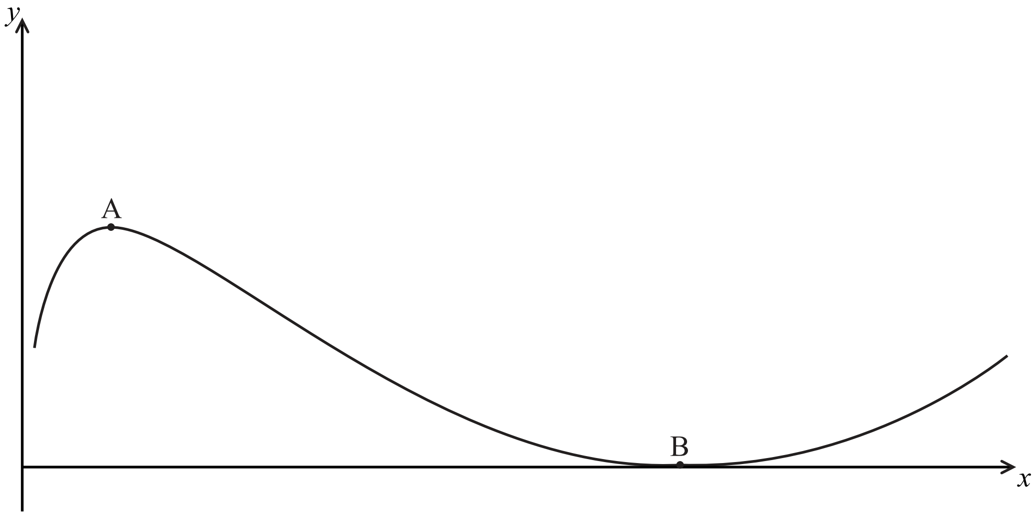
. . . .

. . . .

. . . .

4. *[Maximum mark: 8]*

The diagram shows the graph of the function defined by $y = x(\ln x)^2$ for $x > 0$.



The function has a local maximum at the point A and a local minimum at the point B.

(a) Find the coordinates of the points A and B. *[5 marks]*

(This question continues on the following page)

(Question 4 continued)

(b) Given that the graph of the function has exactly one point of inflexion, find its coordinates. *[3 marks]*

5. *[Maximum mark: 8]*

The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} ae^{-x}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) State the mode of X. *[1 mark]*

. . . .

. . . .

(b) Determine the value of a. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

(This question continues on the following page)

(Question 5 continued)

(c) Find $\mathrm{E}(X)$. *[4 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

6. *[Maximum mark: 7]*

Consider the following equations, where $a, b \in \mathbb{R}$:

$$x + 3y + (a-1)z = 1$$
$$2x + 2y + (a-2)z = 1$$
$$3x + y + (a-3)z = b.$$

(a) If each of these equations defines a plane, show that, for any value of a, the planes do not intersect at a unique point. *[3 marks]*

(This question continues on the following page)

(Question 6 continued)

(b) Find the value of b for which the intersection of the planes is a straight line. *[4 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

7. *[Maximum mark: 7]*

In the triangle PQR, $PQ = 6$, $PR = k$ and $P\hat{Q}R = 30^\circ$.

(a) For the case $k = 4$, find the two possible values of QR. *[4 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

(This question continues on the following page)

(Question 7 continued)

(b) Determine the values of k for which the conditions above define a unique triangle. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

8. *[Maximum mark: 9]*

Consider the curve defined by the equation $x^2 + \sin y - xy = 0$.

(a) Find the gradient of the tangent to the curve at the point (π, π). *[6 marks]*

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

(This question continues on the following page)

(Question 8 continued)

(b) Hence, show that $\tan\theta = \dfrac{1}{1+2\pi}$, where θ is the acute angle between the tangent to the curve at (π, π) and the line $y = x$. *[3 marks]*

9. *[Maximum mark: 6]*

Two boats, A and B, move so that at time t hours, their position vectors, in kilometres, are $\boldsymbol{r}_A = (9t)\boldsymbol{i} + (3-6t)\boldsymbol{j}$ and $\boldsymbol{r}_B = (7-4t)\boldsymbol{i} + (7t-6)\boldsymbol{j}$.

(a) Find the coordinates of the common point of the paths of the two boats. *[4 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

(This question continues on the following page)

(Question 9 continued)

(b) Show that the boats do not collide. *[2 marks]*

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions on the answer sheets provided. Please start each question on a new page.*

10. *[Maximum mark: 19]*

Consider the complex numbers

$$z_1 = 2\sqrt{3}\,\text{cis}\frac{3\pi}{2} \text{ and } z_2 = -1+\sqrt{3}\text{i}.$$

(a) (i) Write down z_1 in Cartesian form.

(ii) Hence determine $(z_1 + z_2)^*$ in Cartesian form. *[3 marks]*

(b) (i) Write z_2 in modulus-argument form.

(ii) Hence solve the equation $z^3 = z_2$. *[6 marks]*

(c) Let $z = r\,\text{cis}\,\theta$, where $r \in \mathbb{R}^+$ and $0 \le \theta < 2\pi$. Find all possible values of r and θ,

(i) if $z^2 = (1+z_2)^2$;

(ii) if $z = -\frac{1}{z_2}$. *[6 marks]*

(d) Find the smallest positive value of n for which $\left(\frac{z_1}{z_2}\right)^n \in \mathbb{R}^+$. *[4 marks]*

*Do **NOT** write solutions on this page.*

11. *[Maximum mark: 18]*

Consider the matrix $\boldsymbol{A}=\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}$, $b\neq 0$.

(a) For $a=2$ and $b=1$, show that $(\boldsymbol{A}^2-3\boldsymbol{A})^2=\boldsymbol{I}$. *[3 marks]*

(b) Find the value of a and the value of b in each of the following cases:

(i) $\boldsymbol{A}\begin{pmatrix} 1 \\ 3 \end{pmatrix}=\begin{pmatrix} 0 \\ -2 \end{pmatrix}$;

(ii) $\boldsymbol{A}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}=\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. *[6 marks]*

(c) Consider the lines l_1 and l_2 defined by the equations $ax+(a-1)y=0$ and $bx+by=1$, respectively, where $b\neq 0$.

(i) Find the coordinates of the intersection point of the lines l_1 and l_2 in terms of a and b.

(ii) Given that the lines are perpendicular, find the coordinates of the point of intersection in terms of b. *[9 marks]*

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 23]*

Consider a function f, defined by $f(x) = \dfrac{x}{2-x}$ for $0 \le x \le 1$.

(a) Find an expression for $(f \circ f)(x)$. *[3 marks]*

Let $F_n(x) = \dfrac{x}{2^n - (2^n - 1)x}$, where $0 \le x \le 1$.

(b) Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x).$$

[8 marks]

(c) Show that $F_{-n}(x)$ is an expression for the inverse of F_n. *[6 marks]*

(d) (i) State $F_n(0)$ and $F_n(1)$.

(ii) Show that $F_n(x) < x$, given $0 < x < 1$, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1}, the x-axis and the line $x = 1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n. *[6 marks]*